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# BRAND SELECTION AND ITS MATRIX STRUCTURE
## —Expansion of its Block Matrix to the Third Order Lag—

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**ABSTRACT**：Focusing on the fact that consumers are apt to buy superior brand when they are accustomed or bored to use current brand, a new method of analysis is introduced. The data set (before buying data and after buying data (for example, former buying data and current buying data)) is stated using a linear model. When this is done, the transition matrix becomes an upper triangular matrix. In this paper, the equation using transition matrix stated by the Block Matrix is expanded to the third order lag and the method is newly re-built. An S-step forecasting model is also introduced. This approach makes it possible to identify brand position in the market and it can be utilized for building useful and effective marketing plan.

**Key Words**: brand selection, matrix structure, brand position, third order lag

## 1．INTRODUCTION

It is often observed that consumers select upper-class brands when they buy after they are bored with their current brand. Suppose that former buying data and current buying data are gathered. Also suppose that upper brand is located upper in the variable array. Then the transition matrix becomes an upper triangular matrix under the supposition that the former buying variables are used as input and the current buying variables are used as output. If the top brand were selected from lower-class brand, skipping intermediate brands, the corresponding part in the upper triangular matrix would be 0. If a transition matrix is identified, an s-step forecasting can be executed. Generalized forecasting matrix components' equations are introduced. Planners for products need to know whether their brand is higher or lower than other products. The matrix structure makes it possible to ascertain this by calculating consumers' activities for brand selection. Thus, this proposed approach makes it possible to execute an effective marketing plan and/or establish a new brand.

Quantitative analysis concerning brand selection has been executed by Yamanaka[5], Takahashi et al.[4]. Yamanaka[5] examined the purchasing process by Markov Transition Probability with the input of advertising expense. Takahashi et al.[4] made analysis by the Brand Selection Probability model using logistics distribution.

In Takeyasu et al. (2008,2011), the matrix structure was analyzed for cases when brand selection was executed toward upper class brands. In this paper, the equation using transition matrix stated by the Block matrix is extended to the third order lag and the method is newly re-built. Such research as this cannot be found as long as searched.

Hereinafter, the matrix structure is clarified for the selection of brand in section 2. Block matrix structure is analyzed when brands are handled in group and $s$-step forecasting is formulated in section 3. Expansion of the model to the third order lag is executed in section 4. Application of this method is extended in section 5.

## 2．BRAND SELECTION AND ITS MATRIX STRUCTURE

### （1）Upper shift of Brand selection

Now, suppose that $x$ is the most upper class brand, $y$ is the second upper class brand, and $z$ is the lowest class brand.

Consumer's behavior of selecting brand might be $z \rightarrow y$ , $y \rightarrow x$ , $z \rightarrow x$ etc. $x \rightarrow z$ might be few.
Suppose that $x$ is current buying variable, and $x_b$ is previous buying variable. Shift to $x$ is executed from $x_b$, $y_b$, or $z_b$.

Therefore, $x$ is stated in the following equation. $a_{ij}$ represents transition probability from $j$-th to $i$-th brand.

$$x = a_{11}x_b + a_{12}y_b + a_{13}z_b$$

Similarly,

$$y = a_{22}y_b + a_{23}z_b$$

and

$$z = a_{33}z_b$$

These are re-written as follows.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix} \tag{1}$$

Set

$$\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}, \qquad \mathbf{X_b} = \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix}$$

then, $\mathbf{X}$ is represented as follows.

$$\mathbf{X} = \mathbf{A}\mathbf{X_b} \tag{2}$$

Here,

$$\mathbf{X} \in \mathbf{R}^3, \mathbf{A} \in \mathbf{R}^{3\times 3}, \mathbf{X_b} \in \mathbf{R}^3$$

$\mathbf{A}$ is an upper triangular matrix.

To examine this, generating following data, which are all consisted by the data in which transition is made from lower brand to upper brand,

$$\mathbf{X^i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \cdots \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{3}$$

$$\mathbf{X_b^i} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \cdots \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{4}$$

$$i = 1 \quad , \quad 2 \quad \cdots \quad N$$

parameter can be estimated using least square method.

Suppose

$$\mathbf{X}^i = \mathbf{A}\mathbf{X_b}^i + \boldsymbol{\varepsilon}^i \tag{5}$$

where $\varepsilon^i = \begin{pmatrix} \varepsilon_1^i \\ \varepsilon_2^i \\ \varepsilon_3^i \end{pmatrix}$ $\qquad i = 1,2,\cdots,N$

and minimize following $J$

$$J = \sum_{i=1}^{N} \boldsymbol{\varepsilon}^{iT} \boldsymbol{\varepsilon}^{i} \rightarrow Min \tag{6}$$

$\hat{\mathbf{A}}$ which is an estimated value of $\mathbf{A}$ is obtained as follows.

$$\hat{\mathbf{A}} = \left( \sum_{i=1}^{N} \mathbf{X}^{i} \mathbf{X}_{\mathbf{b}}^{iT} \right) \left( \sum_{i=1}^{N} \mathbf{X}_{\mathbf{b}}^{i} \mathbf{X}_{\mathbf{b}}^{iT} \right)^{-1} \tag{7}$$

In the data group which are all consisted by the data in which transition is made from lower brand to upper brand, estimated value $\hat{\mathbf{A}}$ should be upper triangular matrix.
If following data which shift to lower brand are added only a few in equation (3) and (4),

$$\mathbf{X}^{i} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{X}_{\mathbf{b}}^{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\hat{\mathbf{A}}$ would contain minute items in the lower part triangle.

**(2) Sorting brand ranking by re-arranging row**

In a general data, variables may not be in order as $x, y, z$. In that case, large and small value lie scattered in $\hat{\mathbf{A}}$. But re-arranging this, we can set in order by shifting row. The large value parts are gathered in upper triangular matrix, and the small value parts are gathered in lower triangular matrix.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \overset{\hat{\mathbf{A}}}{\begin{pmatrix} \bigcirc & \bigcirc & \bigcirc \\ \varepsilon & \bigcirc & \bigcirc \\ \varepsilon & \varepsilon & \bigcirc \end{pmatrix}} \quad \overset{\text{Shifting row}}{\Longleftarrow} \quad \begin{pmatrix} z \\ x \\ y \end{pmatrix} \overset{\hat{\mathbf{A}}}{\begin{pmatrix} \varepsilon & \varepsilon & \bigcirc \\ \bigcirc & \bigcirc & \bigcirc \\ \varepsilon & \bigcirc & \bigcirc \end{pmatrix}} \tag{8}$$

**(3) Matrix structure under the case skipping intermediate class brand is skipped**

It is often observed that some consumers select the most upper class brand from the most lower class brand and skip selecting the intermediate class brand.

We suppose $v, w, x, y, z$ brands (suppose they are laid from upper position to lower position as $v > w > x > y > z$).

In the above case, selection shifts would be

$v \leftarrow z$

$v \leftarrow y$

Suppose they do not shift to $y, x, w$ from $z$, to $x, w$ from $y$, and to $w$ from $x$, then Matrix structure would be as follows.

$$\begin{pmatrix} v \\ w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a_{22} & 0 & 0 & 0 \\ 0 & 0 & a_{33} & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 \\ 0 & 0 & 0 & 0 & a_{55} \end{pmatrix} \begin{pmatrix} v_b \\ w_b \\ x_b \\ y_b \\ z_b \end{pmatrix} \tag{9}$$

## 3. BLOCK MATRIX STRUCTURE IN BRAND GOURPS AND S-STEP FORECASTING

Next, we examine the case in brand groups. Matrices are composed by Block Matrix.

### (1) Brand shift group — in the case of two groups

Suppose brand selection shifts from Corolla class to Mark Ⅱ class in car. In this case, it does not matter which company's car they choose. Thus, selection of cars are executed in a group and brand shift is considered to be done from group to group. Suppose brand groups at time $n$ are as follows.

$\mathbf{X}$ consists of $p$ varieties of goods, and $\mathbf{Y}$ consists of $q$ varieties of goods.

$$\mathbf{X_n} = \begin{pmatrix} x_1^n \\ x_2^n \\ \vdots \\ x_p^n \end{pmatrix}, \qquad \mathbf{Y_n} = \begin{pmatrix} y_1^n \\ y_2^n \\ \vdots \\ y_q^n \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{X_n} \\ \mathbf{Y_n} \end{pmatrix} = \begin{pmatrix} \mathbf{A_{11}}, & \mathbf{A_{12}} \\ \mathbf{0}, & \mathbf{A_{22}} \end{pmatrix} \begin{pmatrix} \mathbf{X_{n-1}} \\ \mathbf{Y_{n-1}} \end{pmatrix} \tag{10}$$

Here,

$$\mathbf{X_n} \in \mathbf{R}^p \ (n = 1,2,\cdots), \quad \mathbf{Y_n} \in \mathbf{R}^q \ (n = 1,2,\cdots), \quad \mathbf{A_{11}} \in \mathbf{R}^{p \times p}, \quad \mathbf{A_{12}} \in \mathbf{R}^{p \times q}, \quad \mathbf{A_{22}} \in \mathbf{R}^{q \times q}$$

Make one more step of shift, then we obtain following equation.

$$\begin{pmatrix} \mathbf{X_n} \\ \mathbf{Y_n} \end{pmatrix} = \begin{pmatrix} \mathbf{A_{11}}^2, & \mathbf{A_{11}}\mathbf{A_{12}} + \mathbf{A_{12}}\mathbf{A_{22}} \\ \mathbf{0}, & \mathbf{A_{22}}^2 \end{pmatrix} \begin{pmatrix} \mathbf{X_{n-2}} \\ \mathbf{Y_{n-2}} \end{pmatrix} \tag{11}$$

Make one more step of shift again, then we obtain following equation.

$$\begin{pmatrix} \mathbf{X_n} \\ \mathbf{Y_n} \end{pmatrix} = \begin{pmatrix} \mathbf{A_{11}}^3, & \mathbf{A_{11}}^2\mathbf{A_{12}} + \mathbf{A_{11}}\mathbf{A_{12}}\mathbf{A_{22}} + \mathbf{A_{12}}\mathbf{A_{22}}^2 \\ \mathbf{0}, & \mathbf{A_{22}}^3 \end{pmatrix} \begin{pmatrix} \mathbf{X_{n-3}} \\ \mathbf{Y_{n-3}} \end{pmatrix} \tag{12}$$

Similarly,

$$\begin{pmatrix} \mathbf{X_n} \\ \mathbf{Y_n} \end{pmatrix} = \begin{pmatrix} \mathbf{A_{11}}^4, & \mathbf{A_{11}}^3\mathbf{A_{12}} + \mathbf{A_{11}}^2\mathbf{A_{12}}\mathbf{A_{22}} + \mathbf{A_{11}}\mathbf{A_{12}}\mathbf{A_{22}}^2 + \mathbf{A_{12}}\mathbf{A_{22}}^3 \\ \mathbf{0}, & \mathbf{A_{22}}^4 \end{pmatrix} \begin{pmatrix} \mathbf{X_{n-4}} \\ \mathbf{Y_{n-4}} \end{pmatrix} \tag{13}$$

$$\begin{pmatrix} \mathbf{X_n} \\ \mathbf{Y_n} \end{pmatrix} = \begin{pmatrix} \mathbf{A_{11}}^5, & \mathbf{A_{11}}^4\mathbf{A_{12}} + \mathbf{A_{11}}^3\mathbf{A_{12}}\mathbf{A_{22}} + \mathbf{A_{11}}^2\mathbf{A_{12}}\mathbf{A_{22}}^2 + \mathbf{A_{11}}\mathbf{A_{12}}\mathbf{A_{22}}^3 + \mathbf{A_{12}}\mathbf{A_{22}}^4 \\ \mathbf{0}, & \mathbf{A_{22}}^5 \end{pmatrix} \begin{pmatrix} \mathbf{X_{n-5}} \\ \mathbf{Y_{n-5}} \end{pmatrix} \tag{14}$$

Finally, we get generalized equation for $s$-step shift as follows.

$$\begin{pmatrix} \mathbf{X_n} \\ \mathbf{Y_n} \end{pmatrix} = \begin{pmatrix} \mathbf{A_{11}}^s, & \mathbf{A_{11}}^{s-1}\mathbf{A_{12}} + \sum_{k=2}^{s-1} \mathbf{A_{11}}^{s-k}\mathbf{A_{12}}\mathbf{A_{22}}^{k-1} + \mathbf{A_{12}}\mathbf{A_{22}}^{s-1} \\ \mathbf{0}, & \mathbf{A_{22}}^s \end{pmatrix} \begin{pmatrix} \mathbf{X_{n-s}} \\ \mathbf{Y_{n-s}} \end{pmatrix} \tag{15}$$

If we replace $n-s \to n, n \to n+s$ in equation (15), we can make $s$-step forecast.

**(2) Brand shift group － in the case of three groups**

Suppose brand selection is executed in the same group or to the upper group, and also suppose that brand position is $x > y > z$ ( $x$ is upper position). Then brand selection transition matrix would be expressed as

$$\begin{pmatrix} \mathbf{X_n} \\ \mathbf{Y_n} \\ \mathbf{Z_n} \end{pmatrix} = \begin{pmatrix} \mathbf{A_{11}}, & \mathbf{A_{12}}, & \mathbf{A_{13}} \\ \mathbf{0}, & \mathbf{A_{22}}, & \mathbf{A_{23}} \\ \mathbf{0}, & \mathbf{0}, & \mathbf{A_{33}} \end{pmatrix} \begin{pmatrix} \mathbf{X_{n-1}} \\ \mathbf{Y_{n-1}} \\ \mathbf{Z_{n-1}} \end{pmatrix} \tag{16}$$

Where

$$\mathbf{X_n} = \begin{pmatrix} x_1^n \\ x_2^n \\ \vdots \\ x_p^n \end{pmatrix}, \qquad \mathbf{Y_n} = \begin{pmatrix} y_1^n \\ y_2^n \\ \vdots \\ y_q^n \end{pmatrix}, \qquad \mathbf{Z_n} = \begin{pmatrix} z_1^n \\ z_2^n \\ \vdots \\ z_r^n \end{pmatrix}$$

Here,

$$\mathbf{X_n} \in \mathbf{R}^p \ (n = 1,2,\cdots), \quad \mathbf{Y_n} \in \mathbf{R}^q \ (n = 1,2,\cdots), \quad \mathbf{Z_n} \in \mathbf{R}^r \ (n = 1,2,\cdots), \quad \mathbf{A_{11}} \in R^{p\times p}, \quad \mathbf{A_{12}} \in R^{p\times q}$$

$$\mathbf{A_{13}} \in R^{p\times r}, \quad \mathbf{A_{22}} \in R^{q\times q}, \quad \mathbf{A_{23}} \in R^{q\times r}, \quad \mathbf{A_{33}} \in R^{r\times r}$$

These are re-stated as

$$\mathbf{W_n} = \mathbf{A}\mathbf{W_{n-1}} \tag{17}$$

where,

$$\mathbf{W_n} = \begin{pmatrix} \mathbf{X_n} \\ \mathbf{Y_n} \\ \mathbf{Z_n} \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} \mathbf{A_{11}}, & \mathbf{A_{12}}, & \mathbf{A_{13}} \\ \mathbf{0}, & \mathbf{A_{22}}, & \mathbf{A_{23}} \\ \mathbf{0}, & \mathbf{0}, & \mathbf{A_{33}} \end{pmatrix}, \qquad \mathbf{W_{n-1}} = \begin{pmatrix} \mathbf{X_{n-1}} \\ \mathbf{Y_{n-1}} \\ \mathbf{Z_{n-1}} \end{pmatrix}$$

Hereinafter, we shift steps as is done in previous section.
In the general description, we state as

$$\mathbf{W_n} = \mathbf{A^{(s)}}\mathbf{W_{n-s}} \tag{18}$$

Here,

$$\mathbf{A^{(s)}} = \begin{pmatrix} \mathbf{A_{11}}^{(s)}, & \mathbf{A_{12}}^{(s)}, & \mathbf{A_{13}}^{(s)} \\ \mathbf{0}, & \mathbf{A_{22}}^{(s)}, & \mathbf{A_{23}}^{(s)} \\ \mathbf{0}, & \mathbf{0}, & \mathbf{A_{33}}^{(s)} \end{pmatrix}, \qquad \mathbf{W_{n-s}} = \begin{pmatrix} \mathbf{X_{n-s}} \\ \mathbf{Y_{n-s}} \\ \mathbf{Z_{n-s}} \end{pmatrix}$$

From definition,

$$\mathbf{A^{(1)}} = \mathbf{A} \tag{19}$$

In the case $s = 2$, we obtain

$$\begin{aligned} \mathbf{A}^{(2)} &= \begin{pmatrix} \mathbf{A_{11}}, & \mathbf{A_{12}}, & \mathbf{A_{13}} \\ \mathbf{0}, & \mathbf{A_{22}}, & \mathbf{A_{23}} \\ \mathbf{0}, & \mathbf{0}, & \mathbf{A_{33}} \end{pmatrix} \begin{pmatrix} \mathbf{A_{11}}, & \mathbf{A_{12}}, & \mathbf{A_{13}} \\ \mathbf{0}, & \mathbf{A_{22}}, & \mathbf{A_{23}} \\ \mathbf{0}, & \mathbf{0}, & \mathbf{A_{33}} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{A}_{11}^2, & \mathbf{A_{11}A_{12}} + \mathbf{A_{12}A_{22}}, & \mathbf{A_{11}A_{13}} + \mathbf{A_{12}A_{23}} + \mathbf{A_{13}A_{33}} \\ \mathbf{0}, & \mathbf{A}_{22}^2, & \mathbf{A_{22}A_{23}} + \mathbf{A_{23}A_{33}} \\ \mathbf{0}, & \mathbf{0}, & \mathbf{A}_{33}^2 \end{pmatrix} \end{aligned} \tag{20}$$

Next, in the case $s = 3$, we obtain

$$\mathbf{A}^{(3)} = \begin{pmatrix} \mathbf{A}_{11}^3, & \mathbf{A}_{11}^2\mathbf{A}_{12} + \mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{22} + \mathbf{A}_{12}\mathbf{A}_{22}^2, & \mathbf{A}_{11}^2\mathbf{A}_{13} + \mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{23} + \mathbf{A}_{11}\mathbf{A}_{13}\mathbf{A}_{33} + \mathbf{A}_{12}\mathbf{A}_{22}\mathbf{A}_{23} + \mathbf{A}_{12}\mathbf{A}_{23}\mathbf{A}_{33} + \mathbf{A}_{13}\mathbf{A}_{33}^2 \\ \mathbf{0}, & \mathbf{A}_{22}^3, & \mathbf{A}_{22}^2\mathbf{A}_{23} + \mathbf{A}_{22}\mathbf{A}_{23}\mathbf{A}_{33} + \mathbf{A}_{23}\mathbf{A}_{33}^2 \\ \mathbf{0}, & \mathbf{0}, & \mathbf{A}_{33}^3 \end{pmatrix} \quad (21)$$

In the case $s = 4$, equations become wide-spread, so we express each Block Matrix as follows.

$$\left.\begin{aligned}
&\mathbf{A}_{11}^{(4)} = \mathbf{A}_{11}^4 \\
&\mathbf{A}_{12}^{(4)} = \mathbf{A}_{11}^3\mathbf{A}_{12} + \mathbf{A}_{11}^2\mathbf{A}_{12}\mathbf{A}_{22} + \mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{22}^2 + \mathbf{A}_{12}\mathbf{A}_{22}^3 \\
&\mathbf{A}_{13}^{(4)} = \mathbf{A}_{11}^3\mathbf{A}_{13} + \mathbf{A}_{11}^2\mathbf{A}_{12}\mathbf{A}_{23} + \mathbf{A}_{11}^2\mathbf{A}_{13}\mathbf{A}_{33} + \mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{22}\mathbf{A}_{23} + \mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{23}\mathbf{A}_{33} + \mathbf{A}_{11}\mathbf{A}_{13}\mathbf{A}_{33}^2 \\
&\qquad + \mathbf{A}_{12}\mathbf{A}_{22}^2\mathbf{A}_{23} + \mathbf{A}_{12}\mathbf{A}_{22}\mathbf{A}_{23}\mathbf{A}_{33} + \mathbf{A}_{12}\mathbf{A}_{23}\mathbf{A}_{33}^2 + \mathbf{A}_{13}\mathbf{A}_{33}^3 \\
&\mathbf{A}_{22}^{(4)} = \mathbf{A}_{22}^4 \\
&\mathbf{A}_{23}^{(4)} = \mathbf{A}_{22}^3\mathbf{A}_{23} + \mathbf{A}_{22}^2\mathbf{A}_{23}\mathbf{A}_{33} + \mathbf{A}_{22}\mathbf{A}_{23}\mathbf{A}_{33}^2 + \mathbf{A}_{23}\mathbf{A}_{33}^3 \\
&\mathbf{A}_{33}^{(4)} = \mathbf{A}_{33}^4
\end{aligned}\right\} \quad (22)$$

In the case $s = 5$, we obtain the following equations similarly.

$$\left.\begin{aligned}
&\mathbf{A}_{11}^{(5)} = \mathbf{A}_{11}^5 \\
&\mathbf{A}_{12}^{(5)} = \mathbf{A}_{11}^4\mathbf{A}_{12} + \mathbf{A}_{11}^3\mathbf{A}_{12}\mathbf{A}_{22} + \mathbf{A}_{11}^2\mathbf{A}_{12}\mathbf{A}_{22}^2 + \mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{22}^3 + \mathbf{A}_{12}\mathbf{A}_{22}^4 \\
&\mathbf{A}_{13}^{(5)} = \mathbf{A}_{11}^4\mathbf{A}_{13} + \mathbf{A}_{11}^3\mathbf{A}_{12}\mathbf{A}_{23} + \mathbf{A}_{11}^3\mathbf{A}_{13}\mathbf{A}_{33} + \mathbf{A}_{11}^2\mathbf{A}_{12}\mathbf{A}_{22}\mathbf{A}_{23} + \mathbf{A}_{11}^2\mathbf{A}_{12}\mathbf{A}_{23}\mathbf{A}_{33} + \mathbf{A}_{11}^2\mathbf{A}_{13}\mathbf{A}_{33}^2 \\
&\qquad + \mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{22}^2\mathbf{A}_{23} + \mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{22}\mathbf{A}_{23}\mathbf{A}_{33} + \mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{23}\mathbf{A}_{33}^2 + \mathbf{A}_{11}\mathbf{A}_{13}\mathbf{A}_{33}^3 \\
&\qquad + \mathbf{A}_{12}\mathbf{A}_{22}^3\mathbf{A}_{23} + \mathbf{A}_{12}\mathbf{A}_{22}^2\mathbf{A}_{23}\mathbf{A}_{33} + \mathbf{A}_{12}\mathbf{A}_{22}\mathbf{A}_{23}\mathbf{A}_{33}^2 + \mathbf{A}_{12}\mathbf{A}_{23}\mathbf{A}_{33}^3 + \mathbf{A}_{13}\mathbf{A}_{33}^4 \\
&\mathbf{A}_{22}^{(5)} = \mathbf{A}_{22}^5 \\
&\mathbf{A}_{23}^{(5)} = \mathbf{A}_{22}^4\mathbf{A}_{23} + \mathbf{A}_{22}^3\mathbf{A}_{23}\mathbf{A}_{33} + \mathbf{A}_{22}^2\mathbf{A}_{23}\mathbf{A}_{33}^2 + \mathbf{A}_{22}\mathbf{A}_{23}\mathbf{A}_{33}^3 + \mathbf{A}_{23}\mathbf{A}_{33}^4 \\
&\mathbf{A}_{33}^{(5)} = \mathbf{A}_{33}^5
\end{aligned}\right\} \quad (23)$$

In the case $s = 6$, we obtain

$$\left.\begin{aligned}
&\mathbf{A}_{11}^{(6)} = \mathbf{A}_{11}^6 \\
&\mathbf{A}_{12}^{(6)} = \mathbf{A}_{11}^5\mathbf{A}_{12} + \mathbf{A}_{11}^4\mathbf{A}_{12}\mathbf{A}_{22} + \mathbf{A}_{11}^3\mathbf{A}_{12}\mathbf{A}_{22}^2 + \mathbf{A}_{11}^2\mathbf{A}_{12}\mathbf{A}_{22}^3 + \mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{22}^4 + \mathbf{A}_{12}\mathbf{A}_{22}^5 \\
&\mathbf{A}_{13}^{(6)} = \mathbf{A}_{11}^5\mathbf{A}_{13} + \mathbf{A}_{11}^4\mathbf{A}_{12}\mathbf{A}_{23} + \mathbf{A}_{11}^4\mathbf{A}_{13}\mathbf{A}_{33} + \mathbf{A}_{11}^3\mathbf{A}_{12}\mathbf{A}_{22}\mathbf{A}_{23} + \mathbf{A}_{11}^3\mathbf{A}_{12}\mathbf{A}_{23}\mathbf{A}_{33} + \mathbf{A}_{11}^3\mathbf{A}_{13}\mathbf{A}_{33}^2 \\
&\qquad + \mathbf{A}_{11}^2\mathbf{A}_{12}\mathbf{A}_{22}^2\mathbf{A}_{23} + \mathbf{A}_{11}^2\mathbf{A}_{12}\mathbf{A}_{22}\mathbf{A}_{23}\mathbf{A}_{33} + \mathbf{A}_{11}^2\mathbf{A}_{12}\mathbf{A}_{23}\mathbf{A}_{33}^2 + \mathbf{A}_{11}^2\mathbf{A}_{13}\mathbf{A}_{33}^3 \\
&\qquad + \mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{22}^3\mathbf{A}_{23} + \mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{22}^2\mathbf{A}_{23}\mathbf{A}_{33} + \mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{22}\mathbf{A}_{23}\mathbf{A}_{33}^2 + \mathbf{A}_{11}\mathbf{A}_{12}\mathbf{A}_{23}\mathbf{A}_{33}^3 + \mathbf{A}_{11}\mathbf{A}_{13}\mathbf{A}_{33}^4 \\
&\qquad + \mathbf{A}_{12}\mathbf{A}_{22}^4\mathbf{A}_{23} + \mathbf{A}_{12}\mathbf{A}_{22}^3\mathbf{A}_{23}\mathbf{A}_{33} + \mathbf{A}_{12}\mathbf{A}_{22}^2\mathbf{A}_{23}\mathbf{A}_{33}^2 + \mathbf{A}_{12}\mathbf{A}_{22}\mathbf{A}_{23}\mathbf{A}_{33}^3 + \mathbf{A}_{12}\mathbf{A}_{23}\mathbf{A}_{33}^4 + \mathbf{A}_{13}\mathbf{A}_{33}^5
\end{aligned}\right\} \quad (24)$$

We get generalized equations for $s$-step shift as follows.

$$\left.\begin{aligned}
&\mathbf{A}_{11}^{(s)} = \mathbf{A}_{11}^{s}\\
&\mathbf{A}_{12}^{(s)} = \mathbf{A}_{11}^{s-1}\mathbf{A}_{12} + \sum_{k=2}^{s-1}\mathbf{A}_{11}^{s-k}\mathbf{A}_{12}\mathbf{A}_{22}^{k-1} + \mathbf{A}_{12}\mathbf{A}_{22}^{s-1}\\
&\mathbf{A}_{13}^{(s)} = \mathbf{A}_{11}^{s-1}\mathbf{A}_{13} + \mathbf{A}_{11}^{s-2}\left(\sum_{k=1}^{2}\mathbf{A}_{1(k+1)}\mathbf{A}_{(k+1)3}\right) + \sum_{j=1}^{s-3}\left[\mathbf{A}_{11}^{S-2-j}\left\{\mathbf{A}_{12}\left(\sum_{k=1}^{j+1}\mathbf{A}_{22}^{j+1-k}\mathbf{A}_{23}\mathbf{A}_{33}^{k-1}\right) + \mathbf{A}_{13}\mathbf{A}_{33}^{j+1}\right\}\right]\\
&\mathbf{A}_{22}^{(s)} = \mathbf{A}_{22}^{s}\\
&\mathbf{A}_{23}^{(s)} = \sum_{K-1}^{s}\mathbf{A}_{22}^{s-k}\mathbf{A}_{23}\mathbf{A}_{33}^{k-1}\\
&\mathbf{A}_{33}^{(s)} = \mathbf{A}_{33}^{s}
\end{aligned}\right\} \tag{25}$$

Expressing them in matrix, it follows.

$$\mathbf{A}^{(S)} = \begin{pmatrix}
\mathbf{A}_{11}^{s}, & \mathbf{A}_{11}^{s-1}\mathbf{A}_{12} + \sum_{k=2}^{s-1}\mathbf{A}_{11}^{s-k}\mathbf{A}_{12}\mathbf{A}_{22}^{k-1} + \mathbf{A}_{12}\mathbf{A}_{22}^{s-1}, & \mathbf{A}_{11}^{s-1}\mathbf{A}_{13} + \mathbf{A}_{11}^{s-2}\left(\sum_{k=1}^{2}\mathbf{A}_{1(k+1)}\mathbf{A}_{(K+1)3}\right) + \sum_{j=1}^{s-3}\left[\mathbf{A}_{11}^{s-2-j}\left\{\mathbf{A}_{12}\left(\sum_{k=1}^{j+1}\mathbf{A}_{22}^{j+1-k}\mathbf{A}_{23}\mathbf{A}_{33}^{k-1}\right) + \mathbf{A}_{13}\mathbf{A}_{33}^{j+1}\right\}\right]\\
\mathbf{0}, & \mathbf{A}_{22}^{s}, & \sum_{k=1}^{s}\mathbf{A}_{22}^{s-k}\mathbf{A}_{23}\mathbf{A}_{33}^{k-1}\\
\mathbf{0}, & \mathbf{0}, & \mathbf{A}_{33}^{s}
\end{pmatrix} \tag{26}$$

Generalizing them to $m$ groups, they are expressed as

$$\begin{pmatrix}\mathbf{X}_{\mathbf{n}}^{(1)}\\ \mathbf{X}_{\mathbf{n}}^{(2)}\\ \vdots\\ \mathbf{X}_{\mathbf{n}}^{(m)}\end{pmatrix} = \begin{pmatrix}\mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1m}\\ \mathbf{A}_{21} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2m}\\ \vdots & \vdots & & \vdots\\ \mathbf{A}_{m1} & \mathbf{A}_{m2} & \cdots & \mathbf{A}_{mm}\end{pmatrix}\begin{pmatrix}\mathbf{X}_{\mathbf{n-1}}^{(1)}\\ \mathbf{X}_{\mathbf{n-1}}^{(2)}\\ \vdots\\ \mathbf{X}_{\mathbf{n-1}}^{(m)}\end{pmatrix} \tag{27}$$

$$\mathbf{X}_{\mathbf{n}}^{(1)} \in R^{k_1},\quad \mathbf{X}_{\mathbf{n}}^{(2)} \in R^{k_2},\ \cdots,\ \mathbf{X}_{\mathbf{n}}^{(m)} \in R^{k_m},\quad \mathbf{A}_{\mathbf{ij}} \in R^{k_i \times k_j}\ (i = 1,\cdots,m)(j = 1,\cdots,m)$$

## 4. Expansion to the third order lag

Expansion of the above stated Block Matrix model to the third order lag is executed in the following method. Here we take three groups case.

Generating Eq.(16) and Eq.(18),we state the model as follows. Here we set P=3.

$$\begin{pmatrix}\mathbf{X}_n\\ \mathbf{Y}_n\\ \mathbf{Z}_n\end{pmatrix} = \begin{pmatrix}\mathbf{A}, & \mathbf{B}, & \mathbf{C}\\ \mathbf{D}, & \mathbf{E}, & \mathbf{F}\\ \mathbf{G}, & \mathbf{H}, & \mathbf{J}\end{pmatrix}\begin{pmatrix}\mathbf{X}_{n-1}\\ \mathbf{Y}_{n-1}\\ \mathbf{Z}_{n-1}\end{pmatrix} \tag{28}$$

Where

$$\mathbf{X}_n = \begin{pmatrix} x_1^n \\ x_2^n \\ x_3^n \end{pmatrix}, \mathbf{Y}_n = \begin{pmatrix} y_1^n \\ y_2^n \\ y_3^n \end{pmatrix}, \mathbf{Z_n} = \begin{pmatrix} z_1^n \\ z_2^n \\ z_3^n \end{pmatrix} \tag{29}$$

Here,

$\mathbf{X}_n \in \mathbf{R}^3 (n = 1,2,\cdots), \mathbf{Y}_n \in \mathbf{R}^3 (n = 1,2,\cdots), \mathbf{Z}_n \in \mathbf{R}^3 (n = 1,2,\cdots), \{\mathbf{A,B,C,D,E,F,G,H,J}\} \in \mathbf{R}^{3\times 3}$

These are re-stated as:

$$\mathbf{W}_n = \mathbf{P}\mathbf{W}_{n-1} \tag{30}$$

$$\mathbf{W}_n = \begin{pmatrix} \mathbf{X}_n \\ \mathbf{Y}_n \\ \mathbf{Z}_n \end{pmatrix} \tag{31}$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{A}, & \mathbf{B}, & \mathbf{C} \\ \mathbf{D}, & \mathbf{E}, & \mathbf{F} \\ \mathbf{G}, & \mathbf{H}, & \mathbf{J} \end{pmatrix} \tag{32}$$

$$\mathbf{W}_{n-1} = \begin{pmatrix} \mathbf{X}_{n-1} \\ \mathbf{Y}_{n-1} \\ \mathbf{Z}_{n-1} \end{pmatrix} \tag{33}$$

If N amount of data exist, we can derive the following the equation similarly as Eq.(5),

$$\mathbf{W}_n^i = \mathbf{P}\mathbf{W}_{n-1}^i + \boldsymbol{\varepsilon}_n^i \left(i = 1,2,\cdots,N\right) \tag{34}$$

and

$$J_n = \sum_{i=1}^{N} \boldsymbol{\varepsilon}_n^{iT} \boldsymbol{\varepsilon}_n^i \rightarrow Min \tag{35}$$

$\hat{\mathbf{P}}$ which is an estimated value of $\mathbf{P}$ is obtained as follows.

$$\hat{\mathbf{P}} = \left( \sum_{i=1}^{N} \mathbf{W}_n^i \mathbf{W}_{n-1}^{iT} \right) \left( \sum_{i=1}^{N} \mathbf{W}_{n-1}^i \mathbf{W}_{n-1}^{iT} \right)^{-1} \tag{36}$$

Now, we expand Eq.(34) to the third order lag model as follows.

$$\mathbf{W}_n^i = \mathbf{P}_1 \mathbf{W}_{n-1}^i + \mathbf{P}_2 \mathbf{W}_{n-2}^i + \mathbf{P}_3 \mathbf{W}^i{}_{n-3} + \boldsymbol{\varepsilon}_n^i \tag{37}$$

Here

$$\mathbf{P}_1 = \begin{pmatrix} \mathbf{A}_1, & \mathbf{B}_1, & \mathbf{C}_1 \\ \mathbf{D}_1, & \mathbf{E}_1, & \mathbf{F}_1 \\ \mathbf{G}_1, & \mathbf{H}_1, & \mathbf{J}_1 \end{pmatrix}, \mathbf{P}_2 = \begin{pmatrix} \mathbf{A}_2, & \mathbf{B}_2, & \mathbf{C}_2 \\ \mathbf{D}_2, & \mathbf{E}_2, & \mathbf{F}_2 \\ \mathbf{G}_2, & \mathbf{H}_2, & \mathbf{J}_2 \end{pmatrix}, \mathbf{P}_3 = \begin{pmatrix} \mathbf{A}_3, & \mathbf{B}_3, & \mathbf{C}_3 \\ \mathbf{D}_3, & \mathbf{E}_3, & \mathbf{F}_3 \\ \mathbf{G}_3, & \mathbf{H}_3, & \mathbf{J}_3 \end{pmatrix} \tag{38}$$

It we set

$$\mathbf{P} = \left(\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3\right) \tag{39}$$

then $\hat{\mathbf{P}}$ can be estimated as follows.

$$\mathbf{P}=\left(\sum_{i=1}^{N}\mathbf{W}_t^i\begin{pmatrix}\mathbf{W}_{t-1}^i\\ \mathbf{W}_{t-2}^i\\ \mathbf{W}_{t-3}^i\end{pmatrix}^T\right)\left(\sum_{i=1}^{N}\begin{pmatrix}\mathbf{W}_{t-1}^i\\ \mathbf{W}_{t-2}^i\\ \mathbf{W}_{t-3}^i\end{pmatrix}\begin{pmatrix}\mathbf{W}_{t-1}^i\\ \mathbf{W}_{t-2}^i\\ \mathbf{W}_{t-3}^i\end{pmatrix}^T\right)^{-1} \tag{40}$$

We further develop this equation as follows.

$$\mathbf{P}=\left(\mathbf{P}_1,\mathbf{P}_2,\mathbf{P}_3\right)$$

$$=\begin{pmatrix}\mathbf{A}_1, & \mathbf{B}_1, & \mathbf{C}_1, & \mathbf{A}_2, & \mathbf{B}_2, & \mathbf{C}_2, & \mathbf{A}_3, & \mathbf{B}_3, & \mathbf{C}_3\\ \mathbf{D}_1, & \mathbf{E}_1, & \mathbf{F}_1, & \mathbf{D}_2, & \mathbf{E}_2, & \mathbf{F}_2, & \mathbf{D}_3, & \mathbf{E}_3, & \mathbf{F}_3\\ \mathbf{G}_1, & \mathbf{H}_1, & \mathbf{J}_1, & \mathbf{G}_2, & \mathbf{H}_2, & \mathbf{J}_2, & \mathbf{G}_3, & \mathbf{H}_3, & \mathbf{J}_3\end{pmatrix}$$

$$=\left(\sum_{i=1}^{N}\mathbf{W}_t^i\mathbf{W}_{t-1}^{iT},\sum_{i=1}^{N}\mathbf{W}_t^i\mathbf{W}_{t-2}^{iT},\sum_{i=1}^{N}\mathbf{W}_t^i\mathbf{W}_{t-3}^{iT}\right)\begin{pmatrix}\sum_{i=1}^{N}\mathbf{W}_{t-1}^i\mathbf{W}_{t-1}^{iT},\sum_{i=1}^{N}\mathbf{W}_{t-1}^i\mathbf{W}_{t-2}^{iT},\sum_{i=1}^{N}\mathbf{W}_{t-1}^i\mathbf{W}_{t-3}^{iT}\\ \sum_{i=1}^{N}\mathbf{W}_{t-2}^i\mathbf{W}_{t-1}^{iT},\sum_{i=1}^{N}\mathbf{W}_{t-2}^i\mathbf{W}_{t-2}^{iT},\sum_{i=1}^{N}\mathbf{W}_{t-2}^i\mathbf{W}_{t-3}^{iT}\\ \sum_{i=1}^{N}\mathbf{W}_{t-3}^i\mathbf{W}_{t-1}^{iT},\sum_{i=1}^{N}\mathbf{W}_{t-3}^i\mathbf{W}_{t-2}^{iT},\sum_{i=1}^{N}\mathbf{W}_{t-3}^i\mathbf{W}_{t-3}^{iT}\end{pmatrix}^{-1}$$

$$=\left(\sum_{i=1}^{N}\begin{pmatrix}\mathbf{x}_t^i\\ \mathbf{y}_t^i\\ \mathbf{z}_t^i\end{pmatrix}\left(\mathbf{x}_{t-1}^{iT},\mathbf{y}_{t-1}^{iT},\mathbf{z}_{t-1}^{iT}\right),\quad \sum_{i=1}^{N}\begin{pmatrix}\mathbf{x}_t^i\\ \mathbf{y}_t^i\\ \mathbf{z}_t^i\end{pmatrix}\left(\mathbf{x}_{t-2}^{iT},\mathbf{y}_{t-2}^{iT},\mathbf{z}_{t-2}^{iT}\right),\quad \sum_{i=1}^{N}\begin{pmatrix}\mathbf{x}_t^i\\ \mathbf{y}_t^i\\ \mathbf{z}_t^i\end{pmatrix}\left(\mathbf{x}_{t-3}^{iT},\mathbf{y}_{t-3}^{iT},\mathbf{z}_{t-3}^{iT}\right)\right)$$

$$\times\begin{pmatrix}\sum_{i=1}^{N}\begin{pmatrix}\mathbf{x}_{t-1}^i\\ \mathbf{y}_{t-1}^i\\ \mathbf{z}_{t-1}^i\end{pmatrix}\left(\mathbf{x}_{t-1}^{iT},\mathbf{y}_{t-1}^{iT},\mathbf{z}_{t-1}^{iT}\right), & \sum_{i=1}^{N}\begin{pmatrix}\mathbf{x}_{t-1}^i\\ \mathbf{y}_{t-1}^i\\ \mathbf{z}_{t-1}^i\end{pmatrix}\left(\mathbf{x}_{t-2}^{iT},\mathbf{y}_{t-2}^{iT},\mathbf{z}_{t-2}^{iT}\right), & \sum_{i=1}^{N}\begin{pmatrix}\mathbf{x}_{t-1}^i\\ \mathbf{y}_{t-1}^i\\ \mathbf{z}_{t-1}^i\end{pmatrix}\left(\mathbf{x}_{t-3}^{iT},\mathbf{y}_{t-3}^{iT},\mathbf{z}_{t-3}^{iT}\right)\\ \sum_{i=1}^{N}\begin{pmatrix}\mathbf{x}_{t-2}^i\\ \mathbf{y}_{t-2}^i\\ \mathbf{z}_{t-2}^i\end{pmatrix}\left(\mathbf{x}_{t-1}^{iT},\mathbf{y}_{t-1}^{iT},\mathbf{z}_{t-1}^{iT}\right), & \sum_{i=1}^{N}\begin{pmatrix}\mathbf{x}_{t-2}^i\\ \mathbf{y}_{t-2}^i\\ \mathbf{z}_{t-2}^i\end{pmatrix}\left(\mathbf{x}_{t-2}^{iT},\mathbf{y}_{t-2}^{iT},\mathbf{z}_{t-2}^{iT}\right), & \sum_{i=1}^{N}\begin{pmatrix}\mathbf{x}_{t-2}^i\\ \mathbf{y}_{t-2}^i\\ \mathbf{z}_{t-2}^i\end{pmatrix}\left(\mathbf{x}_{t-3}^{iT},\mathbf{y}_{t-3}^{iT},\mathbf{z}_{t-3}^{iT}\right)\\ \sum_{i=1}^{N}\begin{pmatrix}\mathbf{x}_{t-3}^i\\ \mathbf{y}_{t-3}^i\\ \mathbf{z}_{t-3}^i\end{pmatrix}\left(\mathbf{x}_{t-1}^{iT},\mathbf{y}_{t-1}^{iT},\mathbf{z}_{t-1}^{iT}\right), & \sum_{i=1}^{N}\begin{pmatrix}\mathbf{x}_{t-3}^i\\ \mathbf{y}_{t-3}^i\\ \mathbf{z}_{t-3}^i\end{pmatrix}\left(\mathbf{x}_{t-2}^{iT},\mathbf{y}_{t-2}^{iT},\mathbf{z}_{t-2}^{iT}\right), & \sum_{i=1}^{N}\begin{pmatrix}\mathbf{x}_{t-3}^i\\ \mathbf{y}_{t-3}^i\\ \mathbf{z}_{t-3}^i\end{pmatrix}\left(\mathbf{x}_{t-3}^{iT},\mathbf{y}_{t-3}^{iT},\mathbf{z}_{t-3}^{iT}\right)\end{pmatrix}^{-1} \tag{41}$$

$$=\begin{pmatrix}\sum_{i=1}^{N}\mathbf{x}_t^i\mathbf{x}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_t^i\mathbf{y}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_t^i\mathbf{z}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_t^i\mathbf{x}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_t^i\mathbf{y}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_t^i\mathbf{z}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_t^i\mathbf{x}_{t-3}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_t^i\mathbf{y}_{t-3}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_t^i\mathbf{z}_{t-3}^{iT}\\ \sum_{i=1}^{N}\mathbf{y}_t^i\mathbf{x}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_t^i\mathbf{y}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_t^i\mathbf{z}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_t^i\mathbf{x}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_t^i\mathbf{y}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_t^i\mathbf{z}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_t^i\mathbf{x}_{t-3}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_t^i\mathbf{y}_{t-3}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_t^i\mathbf{z}_{t-3}^{iT}\\ \sum_{i=1}^{N}\mathbf{z}_t^i\mathbf{x}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_t^i\mathbf{y}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_t^i\mathbf{z}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_t^i\mathbf{x}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_t^i\mathbf{y}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_t^i\mathbf{z}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_t^i\mathbf{x}_{t-3}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_t^i\mathbf{y}_{t-3}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_t^i\mathbf{z}_{t-3}^{iT}\end{pmatrix}$$

$$
=\left(\begin{array}{ccc|ccc|ccc}
\sum_{i=1}^{N}\mathbf{x}_{t-1}^{i}\mathbf{x}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_{t-1}^{i}\mathbf{y}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_{t-1}^{i}\mathbf{z}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_{t-1}^{i}\mathbf{x}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_{t-1}^{i}\mathbf{y}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_{t-1}^{i}\mathbf{z}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_{t-1}^{i}\mathbf{x}_{t-3}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_{t-1}^{i}\mathbf{y}_{t-3}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_{t-1}^{i}\mathbf{z}_{t-3}^{iT} \\
\sum_{i=1}^{N}\mathbf{y}_{t-1}^{i}\mathbf{x}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_{t-1}^{i}\mathbf{y}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_{t-1}^{i}\mathbf{z}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_{t-1}^{i}\mathbf{x}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_{t-1}^{i}\mathbf{y}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_{t-1}^{i}\mathbf{z}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_{t-1}^{i}\mathbf{x}_{t-3}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_{t-1}^{i}\mathbf{y}_{t-3}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_{t-1}^{i}\mathbf{z}_{t-3}^{iT} \\
\sum_{i=1}^{N}\mathbf{z}_{t-1}^{i}\mathbf{x}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_{t-1}^{i}\mathbf{y}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_{t-1}^{i}\mathbf{z}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_{t-1}^{i}\mathbf{x}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_{t-1}^{i}\mathbf{y}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_{t-1}^{i}\mathbf{z}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_{t-1}^{i}\mathbf{x}_{t-3}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_{t-1}^{i}\mathbf{y}_{t-3}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_{t-1}^{i}\mathbf{z}_{t-3}^{iT} \\
\hline
\sum_{i=1}^{N}\mathbf{x}_{t-2}^{i}\mathbf{x}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_{t-2}^{i}\mathbf{y}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_{t-2}^{i}\mathbf{z}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_{t-2}^{i}\mathbf{x}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_{t-2}^{i}\mathbf{y}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_{t-2}^{i}\mathbf{z}_{t-2}^{iT} & \sum_{i=1}^{N}\mathbf{x}_{t-2}^{i}\mathbf{x}_{t-3}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_{t-2}^{i}\mathbf{y}_{t-3}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_{t-2}^{i}\mathbf{z}_{t-3}^{iT} \\
\sum_{i=1}^{N}\mathbf{y}_{t-2}^{i}\mathbf{x}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_{t-2}^{i}\mathbf{y}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_{t-2}^{i}\mathbf{z}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_{t-2}^{i}\mathbf{x}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_{t-2}^{i}\mathbf{y}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_{t-2}^{i}\mathbf{z}_{t-2}^{iT} & \sum_{i=1}^{N}\mathbf{y}_{t-2}^{i}\mathbf{x}_{t-3}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_{t-2}^{i}\mathbf{y}_{t-3}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_{t-2}^{i}\mathbf{z}_{t-3}^{iT} \\
\sum_{i=1}^{N}\mathbf{z}_{t-2}^{i}\mathbf{x}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_{t-2}^{i}\mathbf{y}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_{t-2}^{i}\mathbf{z}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_{t-2}^{i}\mathbf{x}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_{t-2}^{i}\mathbf{y}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_{t-2}^{i}\mathbf{z}_{t-2}^{iT} & \sum_{i=1}^{N}\mathbf{z}_{t-2}^{i}\mathbf{x}_{t-3}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_{t-2}^{i}\mathbf{y}_{t-3}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_{t-2}^{i}\mathbf{z}_{t-3}^{iT} \\
\hline
\sum_{i=1}^{N}\mathbf{x}_{t-3}^{i}\mathbf{x}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_{t-3}^{i}\mathbf{y}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_{t-3}^{i}\mathbf{z}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_{t-3}^{i}\mathbf{x}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_{t-3}^{i}\mathbf{y}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_{t-3}^{i}\mathbf{z}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_{t-3}^{i}\mathbf{x}_{t-3}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_{t-3}^{i}\mathbf{y}_{t-3}^{iT}, & \sum_{i=1}^{N}\mathbf{x}_{t-3}^{i}\mathbf{z}_{t-3}^{iT} \\
\sum_{i=1}^{N}\mathbf{y}_{t-3}^{i}\mathbf{x}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_{t-3}^{i}\mathbf{y}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_{t-3}^{i}\mathbf{z}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_{t-3}^{i}\mathbf{x}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_{t-3}^{i}\mathbf{y}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_{t-3}^{i}\mathbf{z}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_{t-3}^{i}\mathbf{x}_{t-3}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_{t-3}^{i}\mathbf{y}_{t-3}^{iT}, & \sum_{i=1}^{N}\mathbf{y}_{t-3}^{i}\mathbf{z}_{t-3}^{iT} \\
\sum_{i=1}^{N}\mathbf{z}_{t-3}^{i}\mathbf{x}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_{t-3}^{i}\mathbf{y}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_{t-3}^{i}\mathbf{z}_{t-1}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_{t-3}^{i}\mathbf{x}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_{t-3}^{i}\mathbf{y}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_{t-3}^{i}\mathbf{z}_{t-2}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_{t-3}^{i}\mathbf{x}_{t-3}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_{t-3}^{i}\mathbf{y}_{t-3}^{iT}, & \sum_{i=1}^{N}\mathbf{z}_{t-3}^{i}\mathbf{z}_{t-3}^{iT}
\end{array}\right)
$$

We set this as:

$$\mathbf{P}=\left(\mathbf{P}_1,\mathbf{P}_2,\mathbf{P}_3\right)$$

$$
\mathbf{P}=\left(\begin{array}{ccc|ccc|ccc}
\mathbf{K}_1 & \mathbf{K}_2 & \mathbf{K}_3 & \mathbf{L}_1 & \mathbf{L}_2 & \mathbf{L}_3 & \mathbf{M}_1 & \mathbf{M}_2 & \mathbf{M}_3 \\
\mathbf{K}_4 & \mathbf{K}_5 & \mathbf{K}_6 & \mathbf{L}_4 & \mathbf{L}_5 & \mathbf{L}_6 & \mathbf{M}_4 & \mathbf{M}_5 & \mathbf{M}_6 \\
\mathbf{K}_7 & \mathbf{K}_8 & \mathbf{K}_9 & \mathbf{L}_7 & \mathbf{L}_8 & \mathbf{L}_9 & \mathbf{M}_7 & \mathbf{M}_8 & \mathbf{M}_9
\end{array}\right)
$$

$$
\times\left(\begin{array}{ccc|ccc|ccc}
\mathbf{N}_1 & \mathbf{N}_2 & \mathbf{N}_3 & \mathbf{Q}_1 & \mathbf{Q}_2 & \mathbf{Q}_3 & \mathbf{R}_1 & \mathbf{R}_2 & \mathbf{R}_3 \\
\mathbf{N}_4 & \mathbf{N}_5 & \mathbf{N}_6 & \mathbf{Q}_4 & \mathbf{Q}_5 & \mathbf{Q}_6 & \mathbf{R}_4 & \mathbf{R}_5 & \mathbf{R}_6 \\
\mathbf{N}_7 & \mathbf{N}_8 & \mathbf{N}_9 & \mathbf{Q}_7 & \mathbf{Q}_8 & \mathbf{Q}_9 & \mathbf{R}_7 & \mathbf{R}_8 & \mathbf{R}_9 \\
\hline
\mathbf{S}_1 & \mathbf{S}_2 & \mathbf{S}_3 & \mathbf{T}_1 & \mathbf{T}_2 & \mathbf{T}_3 & \mathbf{U}_1 & \mathbf{U}_2 & \mathbf{U}_3 \\
\mathbf{S}_4 & \mathbf{S}_5 & \mathbf{S}_6 & \mathbf{T}_4 & \mathbf{T}_5 & \mathbf{T}_6 & \mathbf{U}_4 & \mathbf{U}_5 & \mathbf{U}_6 \\
\mathbf{S}_7 & \mathbf{S}_8 & \mathbf{S}_9 & \mathbf{T}_7 & \mathbf{T}_8 & \mathbf{T}_9 & \mathbf{U}_7 & \mathbf{U}_8 & \mathbf{U}_9 \\
\hline
\mathbf{V}_1 & \mathbf{V}_2 & \mathbf{V}_3 & \boldsymbol{\alpha}_1 & \boldsymbol{\alpha}_2 & \boldsymbol{\alpha}_3 & \boldsymbol{\beta}_1 & \boldsymbol{\beta}_2 & \boldsymbol{\beta}_3 \\
\mathbf{V}_4 & \mathbf{V}_5 & \mathbf{V}_6 & \boldsymbol{\alpha}_4 & \boldsymbol{\alpha}_5 & \boldsymbol{\alpha}_6 & \boldsymbol{\beta}_4 & \boldsymbol{\beta}_5 & \boldsymbol{\beta}_6 \\
\mathbf{V}_7 & \mathbf{V}_8 & \mathbf{V}_9 & \boldsymbol{\alpha}_7 & \boldsymbol{\alpha}_8 & \boldsymbol{\alpha}_9 & \boldsymbol{\beta}_7 & \boldsymbol{\beta}_8 & \boldsymbol{\beta}_9
\end{array}\right)^{-1}
$$

Then when all consist of the same level shifts or the upper level shifts (suppose $\mathbf{X}>\mathbf{Y}>\mathbf{Z}$),
$\boldsymbol{K}_4,\boldsymbol{K}_7,\boldsymbol{K}_8,\boldsymbol{L}_4,\boldsymbol{L}_7,\boldsymbol{L}_8,\boldsymbol{M}_2,\boldsymbol{M}_7,\boldsymbol{M}_8,\boldsymbol{N}_2,\boldsymbol{N}_3,\boldsymbol{N}_6,\boldsymbol{T}_2,\boldsymbol{T}_3,\boldsymbol{T}_6,\boldsymbol{\beta}_2,\boldsymbol{\beta}_3,\boldsymbol{\beta}_6,\boldsymbol{Q}_4,\boldsymbol{Q}_7,\boldsymbol{Q}_8,\boldsymbol{R}_4,\boldsymbol{R}_7,\boldsymbol{R}_8,\boldsymbol{U}_4,\boldsymbol{U}_7,\boldsymbol{U}_8$
are all 0.

As

$\boldsymbol{N}_4=\boldsymbol{N}_2^T,\quad \boldsymbol{N}_7=\boldsymbol{N}_3^T,\ \boldsymbol{N}_8=\boldsymbol{N}_6^T,\quad \boldsymbol{S}_2=\boldsymbol{Q}_4^T,\boldsymbol{S}_3=\boldsymbol{Q}_7^T,\boldsymbol{S}_6=\boldsymbol{Q}_8^T$

$\boldsymbol{T}_4=\boldsymbol{T}_2^T,\quad \boldsymbol{T}_7=\boldsymbol{T}_3^T,\quad \boldsymbol{T}_8=\boldsymbol{T}_6^T,\quad \boldsymbol{V}_2=\boldsymbol{R}_4^T,\boldsymbol{V}_3=\boldsymbol{R}_7^T,\boldsymbol{V}_6=\boldsymbol{R}_8^T$

$\boldsymbol{\beta}_4=\boldsymbol{\beta}_2^T,\quad \boldsymbol{\beta}_7=\boldsymbol{\beta}_3^T,\quad \boldsymbol{\beta}_8=\boldsymbol{\beta}_6^T,\quad \boldsymbol{\alpha}_2=\boldsymbol{U}_4^T,\boldsymbol{\alpha}_3=\boldsymbol{U}_7^T,\boldsymbol{\alpha}_6=\boldsymbol{U}_8^T$

therefore they are all 0.

$\boldsymbol{N}_1,\boldsymbol{N}_5,\boldsymbol{N}_9,\boldsymbol{T}_1,\boldsymbol{T}_5,\boldsymbol{T}_9,\boldsymbol{\beta}_1,\boldsymbol{\beta}_5,\boldsymbol{\beta}_9$ become diagonal Matrices.

Using a symbol “ * ” as a diagonal matrix, P becomes as follows by using the relation stated above.

$$
\mathbf{P}=\left(\begin{array}{ccc|ccc|ccc}
\mathbf{K}_1, & \mathbf{K}_2, & \mathbf{K}_3, & \mathbf{L}_1, & \mathbf{L}_2, & \mathbf{L}_3, & \mathbf{M}_1, & \mathbf{M}_2, & \mathbf{M}_3 \\
\mathbf{0}, & \mathbf{K}_5, & \mathbf{K}_6, & \mathbf{0}, & \mathbf{L}_5, & \mathbf{L}_6, & \mathbf{0}, & \mathbf{M}_5, & \mathbf{M}_6 \\
\mathbf{0}, & \mathbf{0}, & \mathbf{K}_9, & \mathbf{0}, & \mathbf{0}, & \mathbf{L}_9, & \mathbf{0}, & \mathbf{0}, & \mathbf{M}_9
\end{array}\right)
$$

$$\times \left( \begin{array}{ccc:ccc:ccc} *, & \mathbf{0}, & \mathbf{0}, & \mathbf{N}_1, & \mathbf{N}_2, & \mathbf{N}_3, & \mathbf{R}_1, & \mathbf{R}_2, & \mathbf{R}_3 \\ \mathbf{0}, & *, & \mathbf{0}, & \mathbf{0}, & \mathbf{N}_5, & \mathbf{N}_6, & \mathbf{0}, & \mathbf{R}_5, & \mathbf{R}_6 \\ \mathbf{0}, & \mathbf{0}, & *, & \mathbf{0}, & \mathbf{0}, & \mathbf{N}_9, & \mathbf{0}, & \mathbf{0}, & \mathbf{R}_9 \\ \hdashline \mathbf{S}_1, & \mathbf{0}, & \mathbf{0}, & *, & \mathbf{0}, & \mathbf{0}, & \mathbf{U}_1, & \mathbf{U}_2, & \mathbf{U}_3 \\ \mathbf{S}_4, & \mathbf{S}_5, & \mathbf{0}, & \mathbf{0}, & *, & \mathbf{0}, & \mathbf{0}, & \mathbf{U}_5, & \mathbf{U}_6 \\ \mathbf{S}_7, & \mathbf{S}_8, & \mathbf{S}_9, & \mathbf{0}, & \mathbf{0}, & *, & \mathbf{0}, & \mathbf{0}, & \mathbf{U}_9 \\ \hdashline \mathbf{V}_1, & \mathbf{0}, & \mathbf{0}, & \boldsymbol{\alpha}_1, & \mathbf{0}, & \mathbf{0}, & *, & \mathbf{0}, & \mathbf{0} \\ \mathbf{V}_4, & \mathbf{V}_5, & \mathbf{0}, & \boldsymbol{\alpha}_4, & \boldsymbol{\alpha}_5, & \mathbf{0}, & \mathbf{0}, & *, & \mathbf{0} \\ \mathbf{V}_7, & \mathbf{V}_8, & \mathbf{V}_9, & \boldsymbol{\alpha}_7, & \boldsymbol{\alpha}_8, & \boldsymbol{\alpha}_9, & \mathbf{0}, & \mathbf{0}, & * \end{array} \right)^{-1}$$

## 5. APPLICATION OF THIS METHOD

Consumers' behavior may converge by repeating forecast with above method and total sales of all brands may be reduced. Therefore, the analysis results suggest when and what to put new brand into the market which contribute the expansion of the market.

An application of this method can be considered in the following case. There may arise following case. Consumers and producers do not recognize brand position clearly. But analysis of consumers' behavior let them know their brand position in the market. In such a case, strategic marketing guidance to select brand would be introduced.

Setting in order the brand position of various goods and taking suitable marketing policy, enhancement of sales would be enabled. Setting higher ranked brand, consumption would be promoted.

## 6. CONCLUSION

Consumers often buy products of a higher grade brand as they are accustomed or bored with their current brand.

In this paper, the equation using transition matrix stated by the Block Matrix was expanded to the third order lag and the method was newly re-built. An $s$ -step forecast model was also formulated. Such research as questionnaire investigation of consumers' activity in automobile purchasing should be executed in the near future to verify obtained results. This method would be applicable in such fields as brand bag purchasing, brand wine purchasing, brand dress purchasing etc. This new method should be examined in various cases.

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