A NOTE ON
"ON AUTOMORPHISM GROUPS
OF A CURVE AS LINEAR GROUPS"

Izumi Kuribayashi

Introduction

We work under the situation in [3]. In [2], we
have shown a proposition (Lemma 2.10) on the
induced rotation datum of \( G/G' \) of a rotation
datum of a finite group \( G \) in the case where \( G \)
is cyclic. In this note we shall give a proof of
the proposition in the general case, that is to
say, we shall prove:

PROPOSITION 1. Let \( \lambda \) denote a normal rota-
tion datum of a finite group \( G \). Let \( G' \) be a
normal subgroup of \( G \) and put \( \overline{G} = G/G' \). We
define a new rotation datum \( \overline{\lambda} \) of \( \overline{G} \) of genus
\( g_\overline{\lambda}(\overline{\lambda} \mid \overline{G'}) \) by the relation:

\[
\overline{\lambda}(\overline{A}) = \sum_a \left( \frac{1}{[H_a \cdot G' : H_a]} \right) \times \mu(a)(\overline{\lambda}(A_a)) \quad (\overline{A} \in \overline{G'}),
\]

where, letting \( \{H_a\}_a = \{H \in CY(G) \mid \text{(H mod}
\text{G') = }<A>\}, \) we denote by \( A \) a generator of
\( H \) such that \( (A_a \mod G') = A \). Then we have
the following.

(a) \( \overline{\lambda} \) is normal.

(b) \( \overline{\lambda}(\overline{H}, \overline{\lambda}) = \sum_{\overline{H}} \frac{\overline{\lambda}(H' \cdot \lambda)}{(H \in CY(G))} \),

where \( \{H'\}_a \) denotes an \( Rpv \) \( \{(H \in CY(G) \mid \text{(H mod}
\text{G') = H')}/G\text{-conjugacy}\} \).

(c) \( g_\overline{\lambda}(\overline{\lambda}) = g_\lambda(\lambda) \).

Notation

We use the notation in [3]. See Appendix.

1. Preliminaries

In this section, as a preliminary for the proof
of Proposition 1, we shall analyze a basic prop-
erty of normality (cf. [1, §1, Proposition 4]):

PROPOSITION 2. Let \( \lambda \) be a normal (semi-)
rotation datum of a finite group \( G \). Let \( G' \) de-
note a subgroup of \( G \). Then the (semi-) rotation
datum \( \lambda' \) of \( G' \) induced from \( \lambda \) by
restriction is also normal.

To give a proof of this proposition, we need
the following elementary lemma (cf. [1, §1,
Lemma 5]):

For subgroups \( H \) and \( G' \) of a finite group \( G \),
we put
\( N(G \mid H, G') = \{T \in G \mid T^*(H) \cap G' = H \cap G'\} \).

LEMMA 3. Let \( H \) (resp. \( G' \) ) be a cyclic (resp.
a) subgroup of a finite group \( G \). Put \( H' = 
H \cap G' \). Then we have the following.

(a) \( N_0(H) \subseteq N(G \mid H, G') \subseteq N_0(H') \).

Moreover if \( G' \) is normal in \( G \), then
\( N(G \mid H, G') = N_0(H') \).

(b) \( N(G \mid H, G') \) has a double coset decomposition:

\( N(G \mid H, G') = \bigcup_n H \cdot U_n \cdot N_0(H')(\text{disjoint}) \).

(c) For \( B, B' \subseteq N_0(H') \),
Moreover, if \( G' \) is normal in \( G \), then
\[
f(G \mid H, G') = [N_0(H') : N_0(H')] / [H : H'],
\]
where \( f(G \mid H, G') \) denotes the cardinality of the index set \( \{t\} \) of the decomposition (2).

Proof. (a) This is obvious, because every subgroup of a finite cyclic group is characterized by its order.

(b) It is easy to see following (4) and (5).

(4) The mapping \( N_0(H) \times N(G \mid H, G') \rightarrow N(G \mid H, G') \) given by \((A, T) \mapsto A \cdot T\) is an action on the set \( N(G \mid H, G') \).

(5) The mapping \( N(G \mid H, G') \times N_0(H') \rightarrow N(G \mid H, G') \) given by \((T, B) \mapsto T \cdot B\) is an action on the set \( N(G \mid H, G') \).

It follows from (4) and (5) that \( N(G \mid H, G') \) has such a double coset decomposition as (2).

(c) To prove the "if part" of (3), we assume that \( B' \cdot B^{-1} \) belongs to \( H' \). Since \( U_i \) belongs to \( N_0(H') \), there is an element \( B \) in \( H' \) such that \( U_i \cdot B' \cdot B^{-1} = B'' \cdot U_i \). Then we have \( H \cdot U_i \cdot B'' = H' \cdot B'' \cdot U_i \cdot B = H' \cdot U_i \cdot B \), as desired. To prove the converse, we assume that \( H' \cdot U_i \cdot B = H' \cdot U_i \cdot B'' \). Then \( B' \cdot B^{-1} \) belongs to \( U_i \cdot H' \cdot U_i \), as desired.

The rest of (c) is a consequence of (3) and (a).

Q.E.D.

Proof of Proposition 2. Let \( A' \) denote an element of \( G' \) of order \( n' > 1 \), and let \( H \subseteq CY(G \mid G', H') \) where \( H' = \langle A' \rangle \). (Recall \( CY(G \mid G', H') = \{H \subseteq CY(G) \mid H \cap G' = H'\} \).)

Assume that the set \( \{T_i\} \) is as above. And assume that \( \{T_i\} \) (resp. \( \{T_i\}, \{T_i'\} \)) denotes a complete set of representatives of the cosets in \( H \setminus N_0(H) \) (resp. \( N_0(H) \setminus N(G \mid H, G'), H' \setminus N_0(H') \)). Then we observe by Lemma 3 that

\[
(6) \quad \sum_{i,j}^{t,s} H \cdot T_i \cdot T_j = N(G \mid H, G') = \sum_{t,s}^{t,s} H \cdot U_i \cdot T_j \quad \text{(disjoint)}.
\]

Next let \( A \) denote a generator of \( H \) and put \( n = \#A \). By normality of \( \lambda \) we have an effective element \( a \) of \( R_\ast \) such that \( \lambda \cdot (A) = a \times \sum_{i,j}^{i,j} \gamma_i \cdot ((A \cdot T_i, (A)) \).

Hence we obtain by (6) that

\[
(7) \quad \sum_{i,j}^{i,j} \gamma_i \cdot ((A \cdot T_i, (A))) \times \mu_s(\lambda \cdot (T_i, (A)))
\]

\[
= \mu_s(a) \times \sum_{i,j}^{i,j} \gamma_i \cdot ((A \cdot (T_i, (A))))
\]

\[
= \mu_s(a) \times \sum_{i,j}^{i,j} \gamma_i \cdot ((A \cdot (U_i, (A))))
\]

\[
= \mu_s(a) \times \sum_{i,j}^{i,j} \gamma_i \cdot ((A \cdot (T_i, (A))))\ast (T_i, (A) \ast (U_i, (A)))
\]

\[
= \mu_s(a) \times \sum_{i,j}^{i,j} \gamma_i \cdot ((A \cdot (U_i, (A))))
\]

Finally let \( \{H_\ast\} \) denote an Rpv\( [CY(G \mid G', H') \div G\)-conjugacy]. For each \( H_\ast \) we define \( \{U_\ast\}, \{T_\ast\}_i, A_\ast \) and \( \alpha_\ast \) as above.

Then we have

\[
CY(G \mid G', H') = \{\langle T_\ast \rangle_i (A_\ast) \} \ast_
\]

Hence it follows from (7) and Lemma 1.6 in [2] that

\[
(8) \quad \lambda \cdot (A') = \sum_{a} \sum_{i,j}^{i,j} \gamma_i \cdot ((A \cdot T_i, (A))) \times \mu_s(\lambda \cdot (T_i, (A)))
\]

\[
= \sum_{a} \sum_{i,j}^{i,j} \gamma_i \cdot ((A \cdot (U_i, (A))))
\]

\[
= \sum_{i,j}^{i,j} \gamma_i \cdot ((A \cdot (U_i, (A))))
\]

This implies that \( \lambda \) is normal. Q.E.D.

We shall use, in the next section, the following

COROLLARY 4. Let \( \lambda \) be a rotation datum of a finite group \( G \). Let \( G' \) denote a normal subgroup of \( G \). If \( H' \subseteq CY(G') \), then we have

\[
\ell(H' ; \lambda \mid G') = \sum_{H}^{H} \ell(H ; \lambda) \times f(G \mid H, G')
\]
where H runs over an $\text{Rpv} \{\text{CY}(G, G', H')/G\}$-conjugacy].

Proof. This follows from (8) and Lemma 3.

Q.E.D.

2. Proof of Proposition 1

For $T \subseteq G$, $H \subseteq G$ we denote by $\overline{T}$, $\overline{H}$ the element $(T \mod G')$ of $G$, the subset $(H \mod G')$ of $G$ respectively.

Proof of (a) and (b). First of all we note that (I) defines in fact a well-defined semi-rotation datum of $G$.

We assume that the concerned element $\overline{A}$ of $G$ is induced by $A \in G$. Let $H = \langle A \rangle$ (hence note $H = \langle \overline{A} \rangle$). We are concerned about subgroups in the following figure. Then it is noted that $N_0(H) \cap H \cdot G' = H \cdot (N_0(H) \cap G')$.

\textbf{FIGURE.} \hspace{1cm} (T \subseteq G \mid T^*(H) = \overline{H})

\begin{center}
\begin{tikzpicture}
  \node (H) at (0,0) {$H$};
  \node (H') at (3,0) {$H'$};
  \node (G') at (1.5,-2) {$G'$};
  \node (N0H) at (-2,2) {$N_0(H)$};
  \node (N0H') at (-1.5,-2) {$N_0(H')$};
  \node (T) at (-3,0) {$T$};
  \node (T') at (-3,-2) {$T'$};

  \draw[->] (H) -- (N0H);
  \draw[->] (H) -- (H');
  \draw[->] (H') -- (N0H');
  \draw[->] (T) -- (T');
  \draw[->] (T') -- (N0H');
  \draw[->] (T') -- (N0H);\end{tikzpicture}
\end{center}

Let \{T_i\} (resp. \{T_i\} \cap \{U_i\}) be a complete set of representatives of the cosets in $N_0(H) \cap G' \setminus G'$ (resp. $N_0(H) \cap H \cdot G' \setminus N_0(H)$, $H \cap G' \setminus N_0(H) \cap G'$).

Put $H_{\overline{A}} = (T_{\overline{A}} \cdot U_{\overline{A}})^*(H)$ and let $A_{\overline{A}}$ denote a generator of $H_{\overline{A}}$ such that $\overline{A_{\overline{A}}} = \overline{A}$. Observing that $T_{\overline{A}} \cdot U_{\overline{A}}$ is an $\text{Rpv}\{N_0(H) \setminus \{T \subseteq G \mid T^*(H) = \overline{H}\}\}$, here we note that

\textbf{(9)} \hspace{1cm} \{H_{\overline{A}}\}_{\overline{A}} = \{T^*(H) \mid T \subseteq G, T^*(H) = \overline{H}\}.

Put $\overline{n} = \#A$. Since $(A_{\overline{A}} \cdot (T_{\overline{A}} \cdot U_{\overline{A}})^*(A)) \equiv (A:U_{\overline{A}}^*(A)) \mod n$, first we note that

\begin{align*}
\sum_u \mu_\overline{A}(\lambda_\overline{A}(U_{\overline{A}}^*(A))) & = \sum_u \gamma_\overline{A}(\lambda_\overline{A}(T_{\overline{A}}^*(U_{\overline{A}}^*(A)))) \\
& = \mu_\overline{A}(\lambda_\overline{A}(A)) \times \sum_u \gamma_\overline{A}(\lambda_\overline{A}(U_{\overline{A}}^*(A))) \\
& = [G':N_0(H) \cap G'] \mu_\overline{A}(\lambda_\overline{A}(A)) \times \gamma_\overline{A}([A : U_{\overline{A}}^*(A)]).
\end{align*}

Next by normality of $\lambda$ we have an effective element $a$ of $F_{\lambda A}$ such that $\lambda_a(A) = n(A;G) \times a$. Observing that $T_{\overline{A}} \cdot T_{\overline{A}}$ is an $\text{Rpv} \{H \setminus N_0(H)\}$ and that $(A : T_{\overline{A}}^*(A)) \equiv 1 \mod \overline{a}$, we obtain that

\begin{align*}
\mu_\overline{A}(\lambda_\overline{A}(A)) & = \mu_\overline{A}(a \times \sum_u \gamma_\overline{A}(\lambda_\overline{A}(A)) (A : T_{\overline{A}}^*(A))) \\
& = [N_0(H) \cap G' \setminus H \cap G'] \mu_\overline{A}(a) \times \sum_u \gamma_\overline{A}(\lambda_\overline{A}(A)).
\end{align*}

Thus it follows from (9) and (10) that

\begin{align*}
\sum_u \mu_\overline{A}(\lambda_\overline{A}(A)) & = [G' : H \cap G'] \mu_\overline{A}(a) \times \gamma_\overline{A}(\lambda_\overline{A}(A)) \times \sum_u \gamma_\overline{A}(\lambda_\overline{A}(A)).
\end{align*}

Using this we shall compute $n(A;G) \times \mu_\overline{A}(a)$. Since $\{T_i\}$ is also an $\text{Rpv}\{H \cap G' \setminus N_0(H) \cap G'\}$, we see that $T_{\overline{A}} \cdot U_{\overline{A}}$ is an $\text{Rpv}\{H \cap G' \setminus \{T \subseteq G \mid T^*(H) = \overline{H}\}\}$ hence that $\{T_{\overline{A}} \cdot U_{\overline{A}}\} \subseteq$ is an $\text{Rpv}\{H \cap N_0(H)\}$. Then it follows from (12) that

\begin{align*}
\textbf{(13)} \hspace{1cm} n(A;G) \times \mu_\overline{A}(a) & = [G' : H \cap G'] \mu_\overline{A}(a) \times \sum_u \gamma_\overline{A}(\lambda_\overline{A}(A)) \\
& = \sum_u (1/[H \cap G':H]) \mu_\overline{A}(\lambda_\overline{A}(A)) \\
& = \sum_u (1/[H_{\overline{A}} \cap G':H_{\overline{A}}]) \mu_\overline{A}(\lambda_\overline{A}(A)).
\end{align*}
Finally we shall compute \( \bar{\lambda} \cdot \bar{\lambda} \). To do it, we denote by \( A^{(\beta)} \) a generator of \( H^{(\beta)} \) such that \( \bar{A}^{(\beta)} = \bar{A} \) for each \( \beta \). And for this \( A^{(\beta)} \) we define \( H_{\alpha}^{(\beta)}, A_{\alpha}^{(\beta)} \) and \( \alpha^{(\beta)} \) as above. Then it follows from (9) that
\[
\{H_{\alpha}^{(\beta)}, A_{\alpha}^{(\beta)} \}_{\beta, \alpha} = \{H \in CY(G) \mid \bar{H} = \langle \bar{A} \rangle \}.
\]
Therefore by (1) and (13) we have
\[
\bar{\lambda} \cdot \bar{\lambda} = \frac{\sum_{\beta} \sum_{H_{\alpha}^{(\beta)}} (1/\bar{[H_{\alpha}^{(\beta)} \cdot G \cdot H_{\alpha}^{(\beta)}])}}{\mu_{\bar{\lambda}}(\bar{\lambda} \cdot \bar{A})} = n \cdot \bar{A}, \bar{G} \cdot \mu_{\bar{\lambda}}(\alpha^{(\beta)}).
\]
It is obvious that this implies (a) and (b).

Proof of (c). First we reduce the problem. To do it, we denote an Rpv \([CY(G)/G\text{-conjugacy}]\) by \( [H_{ij}]_{i,j \in I(i)} \), where the indexes \( i,j \) have the property that \( \bar{H}_{ij} \) is \( \bar{G} \)-conjugate to \( \bar{H}_{i1} \) if and only if \( i = i' \). Here we may assume that \( \bar{H}_{ij} = \bar{H}_{ij} \) for any \( j, j' \in J(i) \) and so we denote it by \( H_i \). Also we use the following notation: \( n = \#G \), \( n' = \#G' \), \( \bar{n} = \#G \) and \( g' = g(\lambda \mid G') \). Then, applying A.5 in Appendix to \( \lambda \) and \( \lambda \mid G' \), we have the following relations:
\[
2g(\lambda) - 2 = 2g(\lambda) - 2 = n \cdot \sum_{i,j} \ell(H_{ij}; \lambda \mid G') \left\{ 1 - (1/\#H_{ij}) \right\},
\]
\[
2g(\lambda) - 2 = n' \cdot (2g' - 2) = n' \cdot \sum_{H} \ell(H'; \lambda \mid G') \left\{ 1 - (1/\#H') \right\},
\]
where \( \{H'\}_H \) denotes an Rpv \([CY(G')/G'\text{-conjugacy}]\). These yield that
\[
2g' - 2 - \bar{n}(2g(\lambda) - 2)
\]
\[
= n \cdot \sum_{i,j} \ell(H_{ij}; \lambda \mid G) \left\{ 1 - (1/\#H_{ij}) \right\}
\]
\[
- \oint_{H'} \ell(H'; \lambda \mid G') \left\{ 1 - (1/\#H') \right\}.
\]

On the other hand, observing that \( \{\bar{H}_{ij} \mid \#H_{ij} = 1\} \) is an Rpv \([CY(\bar{G})/\bar{G} \text{-conjugacy}]\), we have by A.5 in Appendix that
\[
2g' - 2 - \bar{n}(2g(\lambda) - 2)
\]
\[
= \bar{n} \cdot \sum_{i,j} \ell(H_{ij}; \lambda \mid G) \left\{ 1 - (1/\#H_{ij}) \right\}.
\]
Thus it suffices to show that
\[
\ell(H_{ij}; \lambda \mid G) \left\{ 1 - (1/\#H_{ij}) \right\} = \bar{n} \cdot \sum_{i,j} \ell(H_{ij}; \lambda \mid G) \left\{ 1 - (1/\#H_{ij}) \right\}.
\]

Next, for subgroups \( H' \) and \( H'' \) of \( G' \), we set:
\[
\chi(H', H'') = \begin{cases} 1 & \text{if } H' \text{ is } G' \text{-conjugate to } H'', \\ 0 & \text{otherwise.} \end{cases}
\]

Put \( H_{ij} = H_{ij} \cdot G' \) and denote an Rpv \([N_{ij} \cdot G' \setminus G]/H_{ij} \cdot G') by \( [U_{ij,k}]_{k \in K(i,j)} \). If we put \( H_{ij,k} = U_{ij,k} \cdot H_i \) and \( H_{ij} = H_{ij} \cdot G' \) \( (k \in K(i,j)) \), then it is easy to see that
\[
\{H_{ij,k}\}_{k \in K(i,j)} \text{ is an Rpv \{the G-conjugacy class containing } H_{ij} \cdot G' \text{-conjugacy}.}
\]

Hence for a fixed element \( H' \) of \( CY(G') \) we obtain a well-defined mapping
\[
CY(G \mid G', H') \to \{H_{ij,k} \mid \chi(H_{ij,k}, H') = 1\}
\]
given by \( H' \mapsto (H_{ij,k} \text{ such that } H_{ij} \text{ is } G \text{-conjugate to } H' \text{ and } H_{ij,k} \text{ is } G' \text{-conjugate to } H') \). Therefore this yields a bijection between an Rpv \([CY(G \mid G', H')/G \text{-conjugacy}]\) and \( \{H_{ij,k} \mid \chi(H_{ij,k}, H') = 1\} \). Observing that \( \ell(\lambda \cdot \lambda) \) and \( f(G \mid \cdot, G') \) are now invariant functions with respect to the relation of \( G \)-conjugacy, we obtain by Corollary 4 that
\[
\ell(H'; \lambda \mid G') \left\{ 1 - (1/\#H') \right\}
\]
\[
= \sum_{i,j,k} \ell(H_{ij,k}; \lambda \mid G') \left\{ 1 - (1/\#H_{ij,k}) \right\}.
\]

On the other hand, observing that \( \{\bar{H}_{ij} \mid \#H_{ij} = 1\} \) is an Rpv \([CY(\bar{G})/\bar{G} \text{-conjugacy}]\), we have by A.5 in Appendix that
\[
\ell(H_{ij}, \lambda \mid G) \left\{ 1 - (1/\#H_{ij}) \right\}
\]
\[
= \sum_{i,j,k} \ell(H_{ij,k}; \lambda \mid G') \left\{ 1 - (1/\#H_{ij,k}) \right\}.
\]

Thus it suffices to show that
\[
\ell(H_{ij}; \lambda \mid G) \left\{ 1 - (1/\#H_{ij}) \right\} = \bar{n} \cdot \sum_{i,j} \ell(H_{ij}; \lambda \mid G) \left\{ 1 - (1/\#H_{ij}) \right\}.
\]

Thus it suffices to show that
\[
\ell(H_{ij}; \lambda \mid G) \left\{ 1 - (1/\#H_{ij}) \right\} = \bar{n} \cdot \sum_{i,j} \ell(H_{ij}; \lambda \mid G) \left\{ 1 - (1/\#H_{ij}) \right\}.
\]

Thus it suffices to show that
\[
\ell(H_{ij}; \lambda \mid G) \left\{ 1 - (1/\#H_{ij}) \right\} = \bar{n} \cdot \sum_{i,j} \ell(H_{ij}; \lambda \mid G) \left\{ 1 - (1/\#H_{ij}) \right\}.
\]

Thus it suffices to show that
\[
\ell(H_{ij}; \lambda \mid G) \left\{ 1 - (1/\#H_{ij}) \right\} = \bar{n} \cdot \sum_{i,j} \ell(H_{ij}; \lambda \mid G) \left\{ 1 - (1/\#H_{ij}) \right\}.
\]
where $H$ runs over an $\text{Rpv}[\text{CY}(G \mid G', H')] \slash G$-conjugacy).

Finally, using (9), we compute the right hand side of (9):

$$(\text{RHS}) \text{ of } (9)$$

$$= \sum_{i,j} \ell(H_{ij} ; \lambda) \{ \bar{n} - (\bar{n} \slash \# H_{ij}) \}$$
$$- \sum_{i} \sum_{j,k} \ell(H_{ij} ; \lambda) \{ f(G \mid H_{ij}, G') \}$$
$$\times \sum_{k} \chi(H_{ij} \cdot H_{k}, H') \times (1 - (1 \slash \# H_{ij}))$$

$$= \sum_{i,j} \ell(H_{ij} ; \lambda) \{ \bar{n} - (\bar{n} \slash \# H_{ij}) \}$$
$$- f(G \mid H_{ij}, G') \times \sum_{k} \chi(H_{ij} \cdot H_{k}, H')$$
$$\times (1 - (1 \slash \# H_{ij})).$$

Here we make two remarks. One is that if $\# H_{ij} \neq 1$, then for each $k \in K(i,j)$ there exists a unique $h$ such that $\chi(H_{ij} \cdot H_{k}, H') = 1$. Hence we have

$$\sum_{k} \chi(H_{ij} \cdot H_{k}, H') \times (1 - (1 \slash \# H_{ij}))$$
$$= \sum_{k} (1 - (1 \slash \# H_{ij})).$$

The other is that

$$\sum_{k} f(G \mid H_{ij}, G')$$
$$= \{ [G : N_{0}(H_{ij}) \cdot G'] \mid [N_{0}(H_{ij}) : N_{0}(H_{ij})]$$
$$\slash [H_{ij} : H_{ij}] \}$$

$$= \{ [G : N_{0}(H_{ij}) \cdot G'] \mid [N_{0}(H_{ij}) \cdot G' : G']$$
$$\slash [H_{ij}, H_{ij}] \}$$

$$= \bar{n} \cdot \# H_{i,j},$$

which follows from Lemma 3. Applying these remarks, we obtain (7) by (9), because

$$(\text{RHS}) \text{ of } (9)$$

$$= \sum_{i,j} \ell(H_{ij} ; \lambda) \{ \bar{n} - (\bar{n} \slash \# H_{ij}) \}$$
$$- \sum_{i} \sum_{j,k} \ell(H_{ij} ; \lambda) \{ f(G \mid H_{ij}, G') \}$$
$$\times \sum_{k} \chi(H_{ij} \cdot H_{k}, H') \times (1 - (1 \slash \# H_{ij}))$$

$$= \bar{n} \cdot \sum_{i,j} \ell(H_{ij} ; \lambda) \{ 1 - (1 \slash \# H_{ij}) \},$$

which is (RHS) of (9).

A. Appendix (cf.[3]) For the sake of completeness we mention at a minimum extent the necessary notations and definitions.

A.1. We denote by $R$, the ring group $\mathcal{O}((Z \slash nZ)^*)$ over $Q$ of the group $(Z \slash nZ)^*$ of units in the ring $Z \slash nZ$. We define a mapping $\gamma : Z \rightarrow R$, as follows: If $(a, n) = 1$, then $\gamma(a) = 1$. Otherwise $\gamma(a)$ be the zero element of $R$. An element of $R$ is expressed uniquely by $a = \sum_{c} c \gamma(a)$, where a runs over the set $I = \{ a \in Z \mid 0 < a < n, (a, n) = 1 \}$. We denote by $\deg : R \rightarrow Q$ the ring homomorphism given by $a \mapsto \sum_{c} c$. We denote by $\mu$, the ring homomorphism of $R$ onto $R$, $(a \mid m)$ and put $R = \bigoplus_{a} R$. We denote by $(A^* : A)$, $\in Z$, the integer $m$ such that $m = k / (k, n) \mod \# A^*$ for an element $A$ of a finite group $G$.

A.2. We define a rotation datum of a finite group $G$. We put

$$\text{CY}(G) = \{ \langle A \rangle \mid A \in G^* \}$$

where $G^* = \{ A \in G \mid \# A \neq 1 \}$, and

$$\text{CY}(G) \triangleright G^* = \{ H \in \text{CY}(G) \mid H \triangleright G^* \}$$

where $G^*$ is a subgroup of $G$. Let $\lambda$ be a mapping of $G$ to $R$. (resp. $G^*$ to $R$) such that $\lambda(A) \in R$, for $A \in G$ (resp. $G^*$). If $\lambda$ satisfies the following properties it is called a rotation datum (resp semi-rotation datum) of $G$:

Let $A,B \in G^*$,

A.1. If $A$ and $B$ are G-conjugate, then $\lambda(A) = \lambda(B)$.

A.2. If $\langle A \rangle = \langle B \rangle$, then $\lambda(A) = \gamma_s(A) \times \lambda(B)$.

A.3. For a (semi-)rotation datum $\lambda$ of $G$ we define a semi-rotation datum $\lambda_*$ of $G$ by the relation

$$\lambda_*(A) = \lambda(A) - \sum_{a} \gamma_s((A : B))$$
$$\times \mu_s((\lambda_*(A)), (A \in G^*))$$

where $A$ denote elements such that $\{ \langle A_1 \rangle \} =$
CY(G | ≅ <A>). We assume that # {<A, }}> a
is equal to the order of the index set {a}. Further, we say that a rotation datum λ is of

\[ \deg(\lambda(1)) = 2 - 2g \] (1 unit element).

We denote this g by g(λ).

A-4. Let λ be a (semi-)rotation datum of G. We put

\[ \ell(A; \lambda) = \deg(\lambda \cdot A)/[N_0(<A>; <A>) \cdot <A>] \]

for A ∈ G" and

\[ \ell(1; \lambda) = 0. \]

Further we define gs(λ) for a rotation datum τ of G by

\[ gs(\lambda) = (1/\#G) \sum_{A \in G} (2 - \deg(\lambda(A))) / 2. \]

A-5. Let λ be a rotation datum of G. Then we have

\[ 2 \cdot gs(\lambda) - 2 = n(2 \cdot gs(\lambda) - 2) \]

\[ + n \sum_k \ell(H_k; \lambda) \cdot \{1 - (1/\#H_k)\}. \]

where n = #G and \( H_k \) denote a complete set of representatives of G-conjugacy classes in
CY(G). (cf.[1, §1. Proposition 1])

A-6. An element of R, is said to be effective if it is a finite sum \( \sum \gamma_r(a_r) \) with \( a_r \in \mathbb{Z} \).

We say that a (semi-)rotation datum of G is normal if for each A ∈ G" there exists an effective element a of R such that

\[ \lambda \cdot (A) = n(A; G) \times a, \]

where \( n(A; G) = \sum_{i} \gamma(a_i)(A: T_i \cdot (A)) \).

Here \( T_i \cdot (A) = T_i^{-1} A T_i \) and \( \{T_i\} \) denotes a complete set of representatives of the cosets in
\( <A> \backslash N_0(<A>) \).

References

[1] Kuribayashi, I., On an algebraization of the
